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Form factors of the XXZ model and the affine quantum group symmetry

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Abstract. We present new expressions of form factors of the XXZ model which satisfy Smirnov's three axioms. These new form factors are obtained by applying the affine quantum group $U_q(\widehat{\mathfrak{sl}_2})$ to the known ones obtained in our previous works. We also find the relations among all the new and known form factors, i.e. all other form factors are obtained by applying $U_q(\widehat{\mathfrak{sl}_2})$ to a singlet form factor.

1. Introduction

In [1] we presented integral formulae of solutions to the quantum Knizhnik–Zamolodchikov (q-KZ) equation [2] of level 0 associated with the vector representation of the affine quantum group $U_q(\widehat{\mathfrak{sl}_2})$. Those solutions satisfy Smirnov's three axioms of form factors [3].

Throughout the study of form factors of the sine-Gordon model, Smirnov [3] found that his three axioms are sufficient conditions of local commutativity of local fields of the model. Smirnov also constructed the space of local fields of the sine-Gordon model [4], from the standpoint of the form factor bootstrap formalism. Smirnov's formulae for form factors of the sine-Gordon model are expressed in terms of the deformed Abelian integrals, or deformed hyperelliptic integrals [5].

Babelon *et al* [6] computed form factors of the restricted sine-Gordon model at the reflectionless point, by quantizing solitons of the model. They also found null vectors of the model [7], which lead to a set of differential equations for form factors.

A form factor was originally defined as a matrix element of a local operator. In this paper, however, we call any vector-valued function a 'form factor' that satisfies Smirnov's three axioms. In this sense, the integral formulae given in [1] are form factors of the XXZ model. Furthermore, we wish to consider the space of form factors of the XXZ model, or solutions of Smirnov's three axioms. Our earlier motivation is the question: Is the space of form factors invariant under the action of the affine quantum group $U_q(\widehat{\mathfrak{sl}_2})$, that is the symmetry of the XXZ model?

Let us consider the spin $\frac{1}{2}$ XXZ model with the nearest-neighbour interaction:

$$H_{XXZ} = -\frac{1}{2}\sum_{n=-\infty}^{\infty}(\sigma^x_{n+1}\sigma^x_n + \sigma^y_{n+1}\sigma^y_n + \Delta\sigma^z_{n+1}\sigma^z_n) \tag{1.1}$$

where $\Delta = (q+q^{-1})/2$ and $-1 < q < 0$.

Let $V = \mathbb{C}^2$ be a vector representation of the affine quantum group $U_q(\widehat{\mathfrak{sl}_2})$. The XXZ Hamiltonian H_{XXZ} formally acts on $\cdots \otimes V \otimes V \otimes \cdots$. This Hamiltonian commutes with

$U_q(\widehat{\mathfrak{sl}_2})$. In [8, 9] the space of states $V^{\otimes\infty}$ was identified with the tensor product of level 1 highest and level -1 lowest representations of $U_q(\widehat{\mathfrak{sl}_2})$.

Since the XXZ model possesses $U_q(\widehat{\mathfrak{sl}_2})$-symmetry, any physical quantities of the model are expected to also possess the same symmetry. The following question is thus natural. Letting $G(\zeta)$ be a $V^{\otimes N}$-valued form factor of the XXZ model, does $\pi_\zeta(y)G(\zeta)$, where $y \in U_q(\widehat{\mathfrak{sl}_2})$, again satisfy Smirnov's axioms?

The answer is as follows. It is not always true that $\pi_\zeta(y)G(\zeta)$ solves Smirnov's axioms even if $G(\zeta)$ does. However, for a form factor that satisfies Smirnov's axioms, there exists $y \in U_q(\widehat{\mathfrak{sl}_2})$ such that $\pi_\zeta(y)G(\zeta)$ again satisfies the axioms.

This paper is organized as follows. In section 2 we summarize the results obtained in the previous paper [1]. In section 3 we present new form factors which satisfy Smirnov's three axioms by applying the affine quantum group $U_q(\widehat{\mathfrak{sl}_2})$ to form factors given in section 2. In section 4 we show the relations among form factors obtained in sections 2 and 3. In section 5 we give some remarks.

2. Integral formula of form factors of the XXZ model

In this section we review Smirnov's three axioms of form factors [3] and the integral formula of form factors of the XXZ model given in [1]. See [10, 1] for explicit expressions of some scalar functions and homogeneous functions below.

For a fixed complex parameter q such that $0 < x = -q < 1$, let U be the affine quantum group $U'_q(\widehat{\mathfrak{sl}_2})$ generated by $e_i, f_i, t_i (i = 0, 1)$ [11]. Set $V = \mathbb{C}v_+ \oplus \mathbb{C}v_-$ and let (π_ζ, V), where $\zeta \in \mathbb{C}\backslash\{0\}$, denote the vector representation of U defined by

$$\begin{aligned} &\pi_\zeta(e_1)(v_+, v_-) = \zeta(0, v_+) \qquad \pi_\zeta(f_1)(v_+, v_-) = \zeta^{-1}(v_-, 0) \\ &\pi_\zeta(t_1)(v_+, v_-) = (qv_+, q^{-1}v_-) \qquad \pi_\zeta(e_0)(v_+, v_-) = \zeta(v_-, 0) \\ &\pi_\zeta(f_0)(v_+, v_-) = \zeta^{-1}(0, v_+) \qquad \pi_\zeta(t_0)(v_+, v_-) = (q^{-1}v_+, qv_-). \end{aligned} \tag{2.1}$$

Let $R(\zeta), S(\zeta) = S_0(\zeta)R(\zeta) \in \mathrm{End}(V \otimes V)$ be the R and S matrix of the XXZ model, where the ratio $S_0(\zeta)$ is a scalar function, which satisfy the intertwining property [11]:

$$X(\zeta_1/\zeta_2)(\pi_{\zeta_1} \otimes \pi_{\zeta_2}) \circ \Delta(y) = (\pi_{\zeta_1} \otimes \pi_{\zeta_2}) \circ \Delta'(y)X(\zeta_1/\zeta_2) \tag{2.2}$$

for $X = R$ or S; Δ and $\Delta' = \sigma \circ \Delta$ are coproducts of U.

For $n \geqslant 0, l \geqslant 0, n + l = N$, let $V^{(nl)}$ be a subspace of $V^{\otimes N}$ such that

$$V^{(nl)} = \bigoplus_{\sum \varepsilon_i = l-n} \mathbb{C}v_{\varepsilon_1} \otimes \cdots \otimes v_{\varepsilon_N}.$$

Here we set N even such that $n \equiv l \bmod 2$, for the simplicity. The odd N cases can be treated similarly.

Let $G^{(nl)}_\varepsilon(\zeta_1, \ldots, \zeta_N) \in V^{(nl)}$ with $\varepsilon = \pm$ be a form factor that satisfies the following three axioms.

Axiom 1. S-matrix symmetry

$$\begin{aligned} &P_{j\,j+1}G^{(nl)}_\varepsilon(\ldots, \zeta_{j+1}, \zeta_j, \ldots) \\ &\qquad = S_{j\,j+1}(\zeta_j/\zeta_{j+1})G^{(nl)}_\varepsilon(\ldots, \zeta_j, \zeta_{j+1}, \ldots) \qquad (1 \leqslant j \leqslant N-1) \end{aligned} \tag{2.3}$$

where $P(x \otimes y) = y \otimes x$ for $x, y \in V$.

Axiom 2. Deformed cyclicity

$$P_{12}\dots P_{N-1N}G_\varepsilon^{(nl)}(\zeta_2,\dots,\zeta_N,\zeta_1 q^{-2}) = \varepsilon r(\zeta_1)D_1 G_\varepsilon^{(nl)}(\zeta_1,\dots,\zeta_N) \tag{2.4}$$

where $r(\zeta)$ is an appropriate scalar function and D_1 is a diagonal operator of the form $D_1 = D\otimes 1\otimes\cdots\otimes 1$. The LHS of (2.4) is the analytic continuation of $P_{12}\dots P_{N-1N}G_\varepsilon^{(nl)}(\zeta_2,\dots,\zeta_N,\zeta_1)$ in the variable ζ_1.

Axiom 3. Annihilation pole condition

The $G_\varepsilon^{(nl)}(\zeta)$ has simple poles at $\zeta_N = \sigma\zeta_{N-1}x^{-1}$ with $\sigma = \pm$, and the residue is given by

$$\mathrm{Res}_{\zeta_N=\sigma\zeta_{N-1}x^{-1}} G_\varepsilon^{(nl)}(\zeta) = \tfrac{1}{2}(I - \varepsilon\sigma^{N+1}r(\sigma\zeta_{N-1}x)D_N \\ \times S_{N-1,N-2}(\zeta_{N-1}/\zeta_{N-2})\cdots S_{N-1,1}(\zeta_{N-1}/\zeta_1))G_{\sigma\varepsilon}^{(n-1l-1)}(\zeta')\otimes u_\sigma \tag{2.5}$$

where $(\zeta) = (\zeta_1,\dots,\zeta_N)$, $(\zeta') = (\zeta_1,\dots,\zeta_{N-2})$, and $u_\sigma = v_+\otimes v_- + \sigma v_-\otimes v_+$. Here D_N is of course a diagonal operator of the form $D_N = 1\otimes\cdots\otimes 1\otimes D$.

Remark 1. Note that the consistency of these three axioms implies the relation $r(\zeta)r(\sigma\zeta x) = \sigma^N$.

Remark 2. Let $|\mathrm{vac}\rangle_i$ $(i = 0, 1)$ be the ground states of the XXZ model, where the subscript i signifies the boundary condition of the ground state. From the standpoint of the vertex operator formalism, $|\mathrm{vac}\rangle_i$ is the canonical element of $V(\Lambda_i)\otimes V(\Lambda_i)^*$, where $V(\Lambda_i)$ is the level 1 highest weight module of the affine quantum group U [8]. Let $\varphi^*(\zeta)$ denote the creation operator. Then the form factor of the local operator $\mathcal{O}$ is given as follows

$$G_i^{(N)}(\zeta_1,\dots,\zeta_N) = {}_i\langle\mathrm{vac}|\mathcal{O}\varphi^*(\zeta_N)\dots\varphi^*(\zeta_1)|\mathrm{vac}\rangle_i.$$

We set $G_\varepsilon^{(N)}(\zeta) = G_0^{(N)}(\zeta) + \varepsilon G_1^{(N)}(\zeta)$ such that the second and third axioms (2.4), (2.5) reduce the closed form in terms of $G_\varepsilon^{(N)}(\zeta)$.

In [10, 1] we constructed a solution of (2.3)–(2.5) as follows. Set $m = n-1$ for $n = l$ and $m = \min(n, l)$ for $n \neq l$, and set $D = D^{(nl)} = q^{-N/2}\,\mathrm{diag}(q^n, q^l)$. Let $\Delta^{(nl)}(x_1,\dots,x_m|z_1,\dots,z_n|z_{n+1},\dots,z_N)$ be a homogeneous polynomial of x's and z's, antisymmetric with respect to x's and symmetric with z_j's $(1\leqslant j\leqslant n)$ and z_i's $(n+1\leqslant i\leqslant N)$, respectively. For such a polynomial, let us define $\langle\Delta^{(nl)}\rangle(x_1,\dots,x_m|\zeta_1,\dots,\zeta_N)\in V^{\otimes N}$ by

$$\langle\Delta^{(nl)}\rangle(x_1,\dots,x_m|\zeta_1,\dots,\zeta_N)^{-\cdots-+\cdots+} = \Delta^{(nl)}(x_1,\dots,x_m|z_1,\dots,z_n|z_{n+1},\dots,z_N) \\ \times\prod_{j=1}^{n}\zeta_j\left(\prod_{i=n+1}^{N}\frac{1}{z_i - z_j\tau^2}\right)$$

$$P_{jj+1}\langle\Delta^{(nl)}\rangle(x_1,\dots,x_m|\dots,\zeta_{j+1},\zeta_j,\dots) \\ = R_{j\,j+1}(\zeta_j/\zeta_{j+1})\langle\Delta^{(nl)}\rangle(x_1,\dots,x_m|\dots,\zeta_j,\zeta_{j+1},\dots)$$

where $z_j = \zeta_j^2 (1\leqslant j\leqslant N)$.

Assume that $n\leqslant l$ for a while. Then an integral formula that solves all the three axioms (2.3)–(2.5) is given as follows

$$G_\varepsilon^{(nl)}(\zeta) = \frac{G_0^{(N)}(\zeta)}{m!}\prod_{\mu=1}^{m}\oint_C\frac{\mathrm{d}x_\mu}{2\pi\mathrm{i}}\Psi_\varepsilon^{(mN)}(x_1,\dots,x_m|\zeta_1,\dots,\zeta_N)\langle\Delta^{(nl)}\rangle(\zeta_1,\dots,\zeta_N) \tag{2.6}$$

where $G_0^{(N)}(\zeta)$ is an appropriate scalar function.

The path of the integral $C = C(z_1,\dots,z_N)$ and the explicit expression of the integral kernel $\Psi_\varepsilon^{(mN)}$ are not important in this paper. See [10, 1] for details. Note that (2.6) ensures the S-matrix symmetry (2.3). The second axiom (2.4) and the third one (2.5) imply the transformation properties [10] and the recursion relation [1] of the kernel $\Psi_{\varepsilon}^{(mN)}(x|\zeta)$, respectively.

The explicit expression of $\Delta^{(nl)}$ is also unimportant in this paper. The essential point concerning $\Delta^{(nl)}$ is the following recursion relations

$$\begin{aligned}\Delta^{(nl)}(x_1,\ldots,x_m|z_1,\ldots,z_n|z_{n+1},\ldots,z_N)|_{z_N=z_nq^{-2}} \\ = \prod_{\mu=1}^{m}(x_\mu - z_nq^{-1})\sum_{\nu=1}^{m}(-1)^{m+\nu}h^{(N-2)}(x_\nu|z_1,\overset{n}{\hat{\ldots}},z_{N-1}) \\ \times\Delta^{(n-1l-1)}(x_1,\overset{\nu}{\hat{\ldots}},x_m|z_1,\ldots,z_{n-1}|z_{n+1},\ldots,z_{N-1}) \qquad (2.7)\end{aligned}$$

where $h^{(N)}(x|z_1,\ldots,z_N)$ is a homogeneous function of degree $N-1$ [10], and the degree condition

$$\deg\Delta^{(nl)} = \binom{m}{2} + nl - n. \qquad (2.8)$$

Note that one can determine $\Delta^{(nl)}$ recursively by using (2.7). From the antisymmetry with respect to x's, $\Delta^{(nl)}$ has the factor $\prod_{\mu<\nu}(x_\mu - x_\nu)$. Hence $\Delta^{(nl)}/\prod_{\mu<\nu}(x_\mu - x_\nu)$ is a polynomial of degree $nl-n$. From the symmetry property with respect to z's, the recursion relation (2.7) gives values of $\Delta^{(nl)}$ at nl points. Thus the polynomial $\Delta^{(nl)}$ can be determined from the initial conditions

$$\Delta^{(0l)} = \Delta^{(11)} = 1 \qquad l > 0. \qquad (2.9)$$

3. Form factors and the action of the affine quantum group

In this section we discuss the transformation properties of the form factors given in the last section under the action of the affine quantum group $U = U_q'(\widehat{\mathfrak{sl}_2})$.

For any $y \in U$, the tensor representation $(\pi_{(\zeta_1,\ldots,\zeta_N)}(y), V^{\otimes N})$ is defined as follows

$$\pi_{(\zeta_1,\ldots,\zeta_N)}(y) = (\pi_{\zeta_1}\otimes\cdots\otimes\pi_{\zeta_N})\circ\Delta^{(N-1)}(y). \qquad (3.1)$$

Let us apply $\pi_{(\zeta_1,\ldots,\zeta_N)}(y)$ to $G_{\varepsilon}^{(nl)}(\zeta_1,\ldots,\zeta_N)$. The action of t_i's are trivial:

$$\pi_\zeta(t_0)G_{\varepsilon}^{(nl)}(\zeta) = q^{n-l}G_{\varepsilon}^{(nl)}(\zeta) \qquad \pi_\zeta(t_1)G_{\varepsilon}^{(nl)}(\zeta) = q^{l-n}G_{\varepsilon}^{(nl)}(\zeta).$$

The action of f_0 is non-trivial but the result is very simple.

Lemma 3.1.

$$\pi_\zeta(f_0)G_{\varepsilon}^{(nl)}(\zeta) = 0. \qquad (3.2)$$

Proof. In order to prove (3.2) it is enough to show

$$\pi_\zeta(f_0)\langle\Delta^{(nl)}\rangle(x|\zeta) = 0. \qquad (3.3)$$

The R-matrix symmetry (2.6) of $\langle\Delta^{(nl)}\rangle(x|\zeta)$ implies that of $\pi_\zeta(f_0)\langle\Delta^{(nl)}\rangle(x|\zeta)$ from the intertwining property (2.2). The arbitrary component of $\pi_\zeta(f_0)\langle\Delta^{(nl)}\rangle(x|\zeta)$ can be expressed in terms of linear combination of the extreme component $(\pi_\zeta(f_0)\langle\Delta^{(nl)}\rangle(x|\zeta_{s(1)},\ldots,\zeta_{s(N)}))^{-\cdots-+\cdots+}$'s, where $s\in\mathfrak{S}_N$. Claim (3.3) thus follows from the fact that $(\pi_\zeta(f_0)\langle\Delta^{(nl)}\rangle(x|\zeta))^{-\cdots-+\cdots+}$ vanishes.

Set $\langle\Delta_{(n-1l+1)}^{(0)}\rangle(x|\zeta) = \pi_\zeta(f_0)\langle\Delta^{(nl)}\rangle(x|\zeta)$. Then $\Delta_{(n-1l+1)}^{(0)}(x_1,\ldots,x_m|z_1,\ldots,z_{n-1}|z_n,\ldots,z_N)$ is proportional to $(\pi_\zeta(f_0)\langle\Delta^{(nl)}\rangle(x|\zeta))^{-\cdots-+\cdots+}$. Thanks to the R-matrix symmetry we obtain

$$\Delta^{(0)}_{(n-1l+1)}(x_1,\ldots,x_m|z_1,\ldots,z_{n-1}|z_n,\ldots,z_N) = \sum_{k=n}^{N}\frac{\prod_{j=1}^{n-1}(z_k-z_jq^{-2})}{\prod_{\substack{i=n\\ i\neq k}}^{N}(z_i-z_k)q^{-1}}\Delta^{(nl)}(x_1,\ldots,x_m|z_1,\ldots,z_{n-1},z_k|z_n,\overset{k}{\hat{\ldots}},z_N). \quad (3.4)$$

Note that the singularity at $z_k = z_i$ in the RHS of (3.4) is spurious, and hence that $\Delta^{(0)}_{(n-1l+1)}$ is a homogeneous polynomial of degree $\binom{m}{2}+(n-1)(l+1)-n$, antisymmetric with respect to x_μ's and symmetric with respect to $\{z_1,\ldots,z_{n-1}\}$ and $\{z_n,\ldots,z_N\}$, respectively. The recursion relation

$$\Delta^{(0)}_{(n-1\,l+1)}(x_1,\ldots,x_m|z_1,\ldots,z_{n-1}|z_n,\ldots,z_N)|_{z_N=z_{n-1}q^{-2}} = \prod_{\mu=1}^{m}(x_\mu-z_nq^{-1})\sum_{\nu=1}^{m}(-1)^{m+\nu}h^{(N-2)}(x_\nu|z_1,\overset{n}{\hat{\ldots}},z_{N-1}) \times\Delta_{(n-2\,l)}(x_1,\overset{\nu}{\hat{\ldots}},x_m|z_1,\ldots,z_{n-2}|z_n,\ldots,z_{N-1})$$

is enough to determine $\Delta^{(0)}_{(n-1l+1)}$ recursively. From the power counting $\Delta^{(0)}_{(0l+1)} = 0$. Thus $\Delta^{(0)}_{(n-1l+1)} = 0$, which implies (3.3). □

Hereafter, we wish to consider $\pi_\zeta(y)G^{(nl)}(\zeta)$ for $y \in U$. For that purpose, let us list the following formulae for $\langle\Delta^{(0)}_{(n+1l-1)}\rangle(x|\zeta) = \pi_\zeta(e_0)\langle\Delta^{(nl)}\rangle(x|\zeta)$, $\langle\Delta^{(1)}_{(n-1l+1)}\rangle(x|\zeta) = \pi_\zeta(e_1)\langle\Delta^{(nl)}\rangle(x|\zeta)$, and $\langle\Delta^{(1)}_{(n+1l-1)}\rangle(x|\zeta) = \pi_\zeta(f_1)\langle\Delta^{(nl)}\rangle(x|\zeta)$:

$$\Delta^{(0)}_{(n+1l-1)}(x_1,\ldots,x_m|z_1,\ldots,z_{n+1}|z_{n+2},\ldots,z_N) = \sum_{k=1}^{n+1}\frac{\prod_{i=n+2}^{N}(z_i-z_kq^{-2})}{\prod_{\substack{j=1\\ j\neq k}}^{n+1}(z_k-z_j)q^{-1}} \times\Delta^{(nl)}(x_1,\ldots,x_m|z_1,\overset{k}{\hat{\ldots}},z_{n+1}|z_k,z_{n+2},\ldots,z_N) \quad (3.5)$$

$$\Delta^{(1)}_{(n-1l+1)}(x_1,\ldots,x_m|z_1,\ldots,z_{n-1}|z_n,\ldots,z_N) = q^{l-n+1}\sum_{k=n}^{N}z_k\frac{\prod_{j=1}^{n-1}(z_k-z_jq^{-2})}{\prod_{\substack{i=n\\ i\neq k}}^{N}(z_i-z_k)q^{-1}} \times\Delta^{(nl)}(x_1,\ldots,x_m|z_1,\ldots,z_{n-1},z_k|z_n,\overset{k}{\hat{\ldots}},z_N) \quad (3.6)$$

$$\Delta^{(1)}_{(n+1l-1)}(x_1,\ldots,x_m|z_1,\ldots,z_{n+1}|z_{n+2},\ldots,z_N) = q^{n-l+1}\sum_{k=1}^{n+1}z_k^{-1}\frac{\prod_{i=n+2}^{N}(z_i-z_kq^{-2})}{\prod_{\substack{j=1\\ j\neq k}}^{n+1}(z_k-z_j)q^{-1}} \times\Delta^{(nl)}(x_1,\ldots,x_m|z_1,\overset{k}{\hat{\ldots}},z_{n+1}|z_k,z_{n+2},\ldots,z_N). \quad (3.7)$$

Expressions (3.5)–(3.7) can be proved in a similar manner as (3.4) was proved.

Let $A_0(\zeta) = G^{(nl)}_\varepsilon(\zeta)$, and $A_j(\zeta) = \pi_\zeta(e_0)A_{j-1}(\zeta)$, where $1 \leqslant j$. Then $A_j(\zeta) = 0$ for $j > l-n$, and $A_{j-1}(\zeta) = (\text{scalar factor})\pi_\zeta(f_0)A_j(\zeta)$ for $1 \leqslant j \leqslant l-n$. These $(l-n+1)$ $\{A_j(\zeta)\}_{0\leqslant j\leqslant l-n}$ form a multiplet. Now the following natural question arises: Do all $A_j(\zeta)$'s satisfy the three axioms (2.3)–(2.5) for a suitable choice of the diagonal operator D?

The S-matrix symmetry is apparently satisfied by any $A_j(\zeta)$. The second and third axioms are unfortunately invalid unless $n = l$. (Since $\pi_\zeta(f_0)G^{(nn)}_\varepsilon(\zeta) = 0 = \pi_\zeta(e_0)G^{(nn)}_\varepsilon(\zeta)$, the case $n = l$ is trivial.)

However, we can consider if $\pi_\zeta(y)G^{(nl)}_\varepsilon(\zeta)$ do satisfy the three axioms, where $y = e_1$ or f_1, because $\pi_\zeta(y)G^{(nl)}_\varepsilon(\zeta)$ for any $y \in U$ always satisfy the first axiom (2.3). We do not have to restrict ourselves to the case $y = e_0$. Actually, $\tilde{G}_\varepsilon^{(n+1l-1)} := \pi_\zeta(f_1)G_\varepsilon^{(nl)}(\zeta)$ solves all the three axioms for $D = q^{-N/2}\,\mathrm{diag}(q^{n-1}, q^{l+1})$.

Theorem 3.2. The vector $\tilde{G}_\varepsilon^{(n+1l-1)} := \pi_\zeta(f_1)G_\varepsilon^{(nl)}(\zeta)$ satisfies (2.3)–(2.5) when we set the diagonal operator $D = D^{(n-1l+1)}$.

Proof. The proof is straightforward. By noticing that the LHS of (2.4) should be interpreted as the analytic continuation in the variable ζ_1, we have

$$\begin{aligned}
&P_{12}\ldots P_{N-1N}\pi_{(\zeta_2,\ldots,\zeta_N,\zeta_1q^{-2})}(f_1)G_\varepsilon^{(nl)}(\zeta_2,\ldots,\zeta_N,\zeta_1q^{-2})\\
&\quad= (\pi_{(\zeta_1q^{-2},\zeta_2,\ldots,\zeta_N)}\circ\Delta'(f_1))P_{12}\ldots P_{N-1N}G_\varepsilon^{(nl)}(\zeta_2,\ldots,\zeta_N,\zeta_1q^{-2})\\
&\quad= \varepsilon r(\zeta_1)(\pi_{\zeta_1q^{-2}}(f_1)\otimes\pi_{(\zeta_2,\ldots,\zeta_N)}(1)+\pi_{\zeta_1q^{-2}}(t_1^{-1})\\
&\qquad\otimes\pi_{(\zeta_2,\ldots,\zeta_N)}(f_1))D_1^{(nl)}G_\varepsilon^{(nl)}(\zeta_1,\ldots,\zeta_N)\\
&\quad= \varepsilon r(\zeta_1)D_1^{(nl)}(q^{n-l}\pi_{\zeta_1}(t_1^{-1}f_1t_1)\\
&\qquad\otimes\pi_{(\zeta_2,\ldots,\zeta_N)}(t_1^{-1}t_1)+\pi_{\zeta_1}(t_1^{-1})\otimes\pi_{(\zeta_2,\ldots,\zeta_N)}(f_1))G_\varepsilon^{(nl)}(\zeta_1,\ldots,\zeta_N)\\
&\quad= \varepsilon r(\zeta_1)(D^{(nl)}t_1^{-1})_1(\pi_{\zeta_1}(f_1)\otimes\pi_{(\zeta_2,\ldots,\zeta_N)}(t_1^{-1})+\pi_{\zeta_1}(1)\\
&\qquad\otimes\pi_{(\zeta_2,\ldots,\zeta_N)}(f_1))G_\varepsilon^{(nl)}(\zeta_1,\ldots,\zeta_N)\\
&\quad= \varepsilon r(\zeta_1)(D^{(nl)}t_1^{-1})_1\tilde{G}_\varepsilon^{(n+1l-1)}(\zeta).
\end{aligned}\tag{3.8}$$

Thus the second axiom is proved.

The third axiom for $\tilde{G}_\varepsilon^{(n+1l-1)}(\zeta)$ can be proved as follows. Since the action of $\pi_\zeta(f_1)$ produces no further singularity of form factors, we have

$$\mathrm{Res}_{\zeta_N=\sigma\zeta_{N-1}x^{-1}}\tilde{G}_\varepsilon^{(n+1l-1)}(\zeta) = \pi_{(\zeta',\zeta_{N-1},\sigma\zeta_{N-1}x^{-1})}(f_1)\mathrm{Res}_{\zeta_N=\sigma\zeta_{N-1}x^{-1}}G_\varepsilon^{(nl)}(\zeta).$$

Note that $\pi_{(\zeta_{N-1},\sigma\zeta_{N-1}x^{-1})}(f_1)u_\sigma = 0$. We also notice that

$$\begin{aligned}
&\pi_{(\zeta',\zeta_{N-1},\sigma\zeta_{N-1}x^{-1})}(f_1)S_{N-1,N-2}(\zeta_{N-1}/\zeta_{N-2})\ldots S_{N-1,1}(\zeta_{N-1}/\zeta_1)\\
&\quad= S_{N-1,N-2}(\zeta_{N-1}/\zeta_{N-2})\ldots S_{N-1,1}(\zeta_{N-1}/\zeta_1)(\pi_{\zeta'}(t_1^{-1})\otimes\pi_{\zeta_{N-1}}(f_1)\otimes\pi_{\zeta_N}(t_1^{-1})\\
&\qquad+\pi_{\zeta'}(f_1)\otimes\pi_{\zeta_{N-1}}(1)\otimes\pi_{\zeta_N}(t_1^{-1})+\pi_{\zeta'}(1)\otimes\pi_{\zeta_{N-1}}(1)\otimes\pi_{\zeta_N}(f_1))
\end{aligned}\tag{3.9}$$

and that the first and the third term of the RHS of (3.9) cancel when they act on $G_{\sigma\varepsilon}^{(n-1l-1)}(\zeta')\otimes D_N^{(nl)}u_\sigma$. Thus we obtain

$$\begin{aligned}
\mathrm{Res}_{\zeta_N=\sigma\zeta_{N-1}x^{-1}}\tilde{G}_\varepsilon^{(n+1l-1)}(\zeta) &= \tfrac{1}{2}(I-\varepsilon\sigma^{N+1}r(\sigma\zeta_{N-1}x)(t_1^{-1}D^{(nl)})_N\\
&\quad\times S_{N-1,N-2}(\zeta_{N-1}/\zeta_{N-2})\ldots S_{N-1,1}(\zeta_{N-1}/\zeta_1))\tilde{G}_{\sigma\varepsilon}^{(n-1l-1)}(\zeta')\otimes u_\sigma
\end{aligned}$$

that implies the third axiom with $D = D^{(n-1l+1)}$. □

Note that the diagonal operator for $\tilde{G}_\varepsilon^{(n+1l-1)}(\zeta)$ is $D^{(n-1l+1)}$ but not $D^{(n+1l-1)}$, so that $\pi_\zeta(f_1)\tilde{G}_\varepsilon^{(n+1l-1)}(\zeta)$ no longer satisfies the second and the third axioms (2.4), (2.5).

4. Relations among form factors of the XXZ model

In this section we shall find further relations among $G_\varepsilon^{(nl)}(\zeta)$'s and $\tilde{G}_\varepsilon^{(n+1l-1)}(\zeta)$'s.

When $N = 2n$ we have the following simple relation between $G_\varepsilon^{(nn)}(\zeta)$ and $G_\varepsilon^{(n-1n+1)}(\zeta)$.

Proposition 4.1.

$$G_\varepsilon^{(n-1n+1)}(\zeta) = (-1)^n q^{-n-1}\pi_\zeta(e_1)G_\varepsilon^{(nn)}(\zeta).\tag{4.1}$$

Proof. By putting $n{=}l$ it follows from (2.7) and (3.6) that $\Delta^{(1)}_{(n-1n+1)}$ and $(-1)^n q^{n+1}\Delta^{(n-1n+1)}$ have the same recursion relation and the same initial condition, and thus the two are the same. Since the integral kernel $\Psi^{(mN)}_{\varepsilon} = \Psi^{(n-1N)}_{\varepsilon}$ is also common for $G^{(nn)}_{\varepsilon}(\zeta)$ and $G^{(n-1n+1)}_{\varepsilon}(\zeta)$, we obtain (4.1). □

Two homogeneous polynomial $\Delta^{(1)}_{(n-1n+1)}$ and $\Delta^{(n-1n+1)}$ coincide up to a constant factor as shown in proposition 4.1. The relation $\Delta^{(1)}_{(n-1l+1)}$ and $\Delta^{(n-1l+1)}$ for $n < l$ is not so simple. In order to establish the relation, let us introduce the symbol $\cong$ as follows. We denote $A(x_1,\ldots,x_m|z_1,\ldots,z_N) \cong B(x_1,\ldots,x_m|z_1,\ldots,z_N)$ when

$$\prod_{\mu=1}^{m}\oint_C \frac{\mathrm{d}x_\mu}{2\pi\mathrm{i}}\Psi^{(mN)}_{\varepsilon}(x|\zeta)A(x|z) = \prod_{\mu=1}^{m}\oint_C \frac{\mathrm{d}x_\mu}{2\pi\mathrm{i}}\Psi^{(mN)}_{\varepsilon}(x|\zeta)B(x|z).$$

Then the following relations hold.

Proposition 4.2.

$$\begin{aligned}\Delta^{(1)}_{(n-1l+1)}&(x_1,\ldots,x_n|z_1,\ldots,z_{n-1}|z_n,\ldots,z_N)\\ &\cong n(l-n+2)(-1)^{l-n}q^{l+1}(1-q^{-2(l-n)})\prod_{j=1}^{n-1}(x_n - z_j q^{-1})\\ &\quad\times\Delta^{(n-1l+1)}(x_1,\ldots,x_{n-1}|z_1,\ldots,z_{n-1}|z_n,\ldots,z_N).\end{aligned} \tag{4.2}$$

Proof. Relation (4.2) follows from the antisymmetry of x's, other than the recursion relation and the initial condition of $\Delta^{(nl)}$.

When $l > n = 1 (N = l+1)$, by using (3.6) we have

$$\Delta^{(1)}_{(0N)}(x_1,\ldots,x_n|\,|z_1,\ldots,z_N) = q^l\sum_{k=1}^{N} z_k\frac{\Delta^{(1l)}(x_1,\ldots,x_n|z_k|z_1,\overset{\hat{k}}{\ldots},z_N)}{\prod_{\substack{i=1\\ i\neq k}}^{N}(z_i - z_k)q^{-1}}. \tag{4.3}$$

The RHS of (4.3) is a constant because $\deg\Delta^{(1)}_{(0N)} = 0$. In order to determine this constant, we substitute the explicit expression of $\Delta^{(1l)}$ [10] and put $x_1 = 0$. Then we have

$$\Delta^{(1)}_{(0N)}(x_1||z_1,\ldots,z_N) = N(-q)^{l+1}(1-q^{-2(l-1)}). \tag{4.4}$$

When $z_N = z_{n-1}q^{-2}$ in (3.6) we have

$$\begin{aligned}\Delta^{(1)}_{(n-1l+1)}&(x_1,\ldots,x_n|z_1,\ldots,z_{n-1}|z_n,\ldots,z_N)|_{z_N=z_{n-1}q^{-2}} = (-q)\prod_{\mu=1}^{n}(x_\mu - z_{n-1}q^{-1})\\ &\times\left\{\sum_{\mu=1}^{n-1}(-1)^{n+\mu}h(x_\mu)\Delta^{(1)}_{(n-2l)}(x_1,\overset{\hat{\mu}}{\ldots},x_{n-1},x_n|z_1,\ldots,z_{n-2}|z_n,\ldots,z_{N-1})\right.\\ &\left.+h(x_n)\Delta^{(1)}_{(n-2l)}(x_1,\ldots,x_{n-1}|z_1,\ldots,z_{n-2}|z_n,\ldots,z_{N-1})\right\}\\ &\cong(-q)\prod_{\mu=1}^{n}(x_\mu - z_{n-1}q^{-1})\left\{\sum_{\mu=1}^{n-1}(-1)^{n+\mu}h(x_\mu)\right.\\ &\times\Delta^{(1)}_{(n-2l)}(x_1,\overset{\hat{\mu}}{\ldots},x_{n-1},x_n|z_1,\ldots,z_{n-2}|z_n,\ldots,z_{N-1}) + \frac{1}{n-1}\\ &\left.\times\sum_{\mu=1}^{n-1}(-1)^{n+\mu}h(x_\mu)\Delta^{(1)}_{(n-2l)}(x_1,\overset{\hat{\mu}}{\ldots},x_{n-1},x_n|z_1,\ldots,z_{n-2}|z_n,\ldots,z_{N-1})\right\}\end{aligned}$$

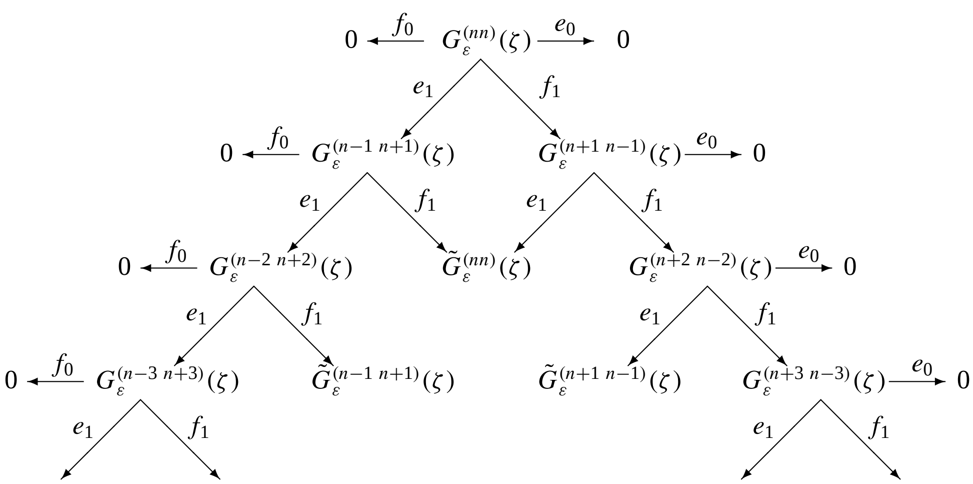
$$\cong (1+\frac{1}{n-1})\times(-q)(n-1)(l-n+2)(-1)^{l-n}q^{l}(1-q^{-2(l-n)})$$
$$\times\prod_{\mu=1}^{n}(x_\mu - z_{n-1}q^{-1})\sum_{\mu=1}^{n-1}(-1)^{n+\mu}h(x_\mu)\prod_{j=1}^{n-2}(x_n - z_j q^{-1})$$
$$\times\Delta^{(n-2l)}(x_1,\overset{\mu}{\hat{\dots}},x_{n-1}|z_1,\dots,z_{n-2}|z_n,\dots,z_{N-1})$$
$$= n(l-n+2)(-1)^{l-n}q^{l+1}(1-q^{-2(l-n)})\prod_{j=1}^{n-1}(x_n - z_j q^{-1})\prod_{\mu=1}^{n-1}(x_\mu - z_{n-1}q^{-1})$$
$$\times\sum_{\mu=1}^{n-1}(-1)^{n-1+\mu}h(x_\mu)\Delta^{(n-2l)}(x_1,\overset{\mu}{\hat{\dots}},x_{n-1}|z_1,\dots,z_{n-2}|z_n,\dots,z_{N-1})$$
$$= (\text{RHS of (4.2)})|_{z_N=z_{n-1}q^{-2}} \tag{4.5}$$

where we use the antisymmetric property with respect to x's, and the assumption of the induction, in the second and third equality, respectively. Equation (4.2) follows from (4.4) and (4.5). □

Until now, we discussed the case $n \leqslant l$. Let us construct $G_\varepsilon^{(nl)}(\zeta)$ with $n > l$ from $G_\varepsilon^{(nn)}(\zeta)$, the spin 0 sector of form factors. Define $G_\varepsilon^{(n+1n-1)}(\zeta) = \pi_\zeta(f_1)G_\varepsilon^{(nn)}(\zeta)$. Then $G_\varepsilon^{(n+1n-1)}(\zeta)$ also satisfies the three axioms with $D = D^{(n-1n+1)}$. By applying f_1 successively, we can obtain $G_\varepsilon^{(n+kn-k)}(\zeta)$ for $n = 1,\dots,n$, just like we constructed $G_\varepsilon^{(n-kn+k)}(\zeta)$ from $G_\varepsilon^{(nn)}(\zeta)$ by applying e_1 successively. As for $G_\varepsilon^{(nl)}(\zeta)$ with $n > l$, $\pi_\zeta(e_0)G_\varepsilon^{(nl)}(\zeta) = 0$ holds. The proof is easy if you notice that $\pi_\zeta(e_0)G_\varepsilon^{(nn)}(\zeta) = 0$ and $[e_0, f_1] = 0$.

Note that $G_\varepsilon^{(nl)}(\zeta)$ with $n > l$ is a form factor; i.e. $G_\varepsilon^{(nl)}(\zeta)$ satisfies the three axioms of form factors with $D = D^{(ln)}$. You can also show that $\tilde{G}_\varepsilon^{(n-1l+1)}(\zeta) := \pi_\zeta(e_1)G_\varepsilon^{(nl)}(\zeta)$ again satisfies the three axioms with $D = D^{(l-1n+1)}$.

Let us summarize the relations obtained until now.



It is evident from this relationship that $G_\varepsilon^{(n-kn+k)}(\zeta)$ and $\tilde{G}_\varepsilon^{(n-kn+k)}(\zeta)$ $(-n \leqslant k \leqslant n)$ can be obtained from $G_\varepsilon^{(nn)}(\zeta)$ by applying e_1 and f_1 in an appropriate order. We again notice that $\pi_\zeta(f_0)G_\varepsilon^{(nn)}(\zeta) = \pi_\zeta(e_0)G_\varepsilon^{(nn)}(\zeta) = 0$.

We naturally have a form factor $F_\varepsilon^{(nn)}(\zeta)$ that belongs to $V^{(nn)}$ such that $\pi_\zeta(f_1)F_\varepsilon^{(nn)}(\zeta) = \pi_\zeta(e_1)F_\varepsilon^{(nn)}(\zeta) = 0$. We can obtain $F_\varepsilon^{(nn)}(\zeta)$ from $G_\varepsilon^{(nn)}(\zeta)$ by a simple transformation.

If $G(\zeta)$ solves the three axioms of form factors with the diagonal operator D, then $F(\zeta) = (\sigma^x)^{\otimes N}G(\zeta)$ solves them with the diagonal operator $\sigma^x D$. Hence $F_\varepsilon^{(ln)}(\zeta) := (\sigma^x)^{\otimes N}G_\varepsilon^{(nl)}(\zeta)$ and $\tilde{F}_\varepsilon^{(ln)}(\zeta) := (\sigma^x)^{\otimes N}\tilde{G}_\varepsilon^{(nl)}(\zeta)$ are also form factors of the XXZ model. We can further show that $\pi_\zeta(f_1)F_\varepsilon^{(nn)}(\zeta) = \pi_\zeta(e_1)F_\varepsilon^{(nn)}(\zeta) = 0$, and $\pi_\zeta(f_1)F_\varepsilon^{(ln)}(\zeta) = 0$ for $n < l$, $\pi_\zeta(e_1)F_\varepsilon^{(nn)}(\zeta) = 0$ for $n > l$.

Sum up the results obtained in this paper: For fixed $n < l$, we find eight form factors which belong to $V^{(nl)}$-sector; i.e. $G_\varepsilon^{(nl)}(\zeta)$, $\tilde{G}_\varepsilon^{(nl)}(\zeta)$, $F_\varepsilon^{(nl)}(\zeta)$ and $\tilde{F}_\varepsilon^{(nl)}(\zeta)$, where $\varepsilon = \pm$. Since we have had $G_\varepsilon^{(nl)}(\zeta)$ only when we fix $n < l$ at the stage of [1], we get four times solutions of the three axioms of form factor in this work.

5. Concluding remarks

In this paper, we have constructed new integral expressions of form factors of the XXZ model, by applying $U_q(\widehat{\mathfrak{sl}_2})$ to the form factors obtained in [1]. The axioms for the form factor $G_\varepsilon^{(nl)}(\zeta)$ with the diagonal operator $D = D^{(nl)}$ reduces those for the form factor $\tilde{G}_\varepsilon^{(n+1l-1)}(\zeta)$ with $D = D^{(n-1l+1)}$ after the action of f_1 when $n < l$; whereas the axioms for $G_\varepsilon^{(nl)}(\zeta)$ with $D = D^{(ln)}$ reduces those for $\tilde{G}_\varepsilon^{(n-1l+1)}(\zeta)$ with $D = D^{(l-1n+1)}$ after the action of e_1 when $n > l$.

The spin 0 form factor $G^{(nn)}(\zeta)$ is a kind of singlet because $\pi_\zeta(f_0)G^{(nn)}(\zeta) = \pi_\zeta(e_0)G^{(nn)}(\zeta) = 0$. In the earlier stage of this work, our goal was to decompose the space of form factors of the XXZ model into infinitely many multiplets of $U_q(\mathfrak{sl}_2)$. Although $G_\varepsilon^{(nl)}(\zeta)$ satisfies three axioms (2.3)–(2.5) and $\pi_\zeta(f_0)G_\varepsilon^{(nl)}(\zeta) = 0$ when $n < l$, $\pi_\zeta(e_0)G_\varepsilon^{(nl)}(\zeta)$ no longer satisfies (2.3)–(2.5). For example, by similar manipulation in (3.8) we have

$$\begin{aligned} P_{12}\ldots P_{N-1N}\pi_{(\zeta_2,\ldots,\zeta_N,\zeta_1 q^{-2})}(e_0)G_\varepsilon^{(nl)}(\zeta_2,\ldots,\zeta_N,\zeta_1 q^{-2}) \\ = \varepsilon r(\zeta_1)D_1^{(n+1l-1)}(q^{2(n-l)}\pi_{\zeta_1}(e_0)\otimes\pi_{(\zeta_2,\ldots,\zeta_N)}(1) + \pi_{\zeta_1}(t_0) \\ \otimes\pi_{(\zeta_2,\ldots,\zeta_N)}(e_0))G_\varepsilon^{(nl)}(\zeta). \end{aligned} \tag{5.1}$$

The RHS of (5.1) reduces to $\pi_\zeta(e_0)G_\varepsilon^{(nl)}(\zeta)$ up to constant at the limit $q \to -1$, which corresponds the XXX model limit.

The XXX model has the Yangian $Y(\mathfrak{sl}_2)$-symmetry. The Yangian $Y(\mathfrak{sl}_2)$ is the minimal quantum group which includes the universal enveloping algebra $U(\mathfrak{sl}_2)$ as a sub-Hopf algebra. Since $U(\mathfrak{sl}_2)$ has the symmetric coproduct unlike $U_q(\mathfrak{sl}_2)$, we may be possible to decompose the space of form factors of the XXX model into infinitely many multiplet of $U(\mathfrak{sl}_2)$. We hope that it is fruitful to consider the XXX model and to find some relations among form factors of the model as obtained in this paper.

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References

[1] Kojima T, Miki K and Quano Y-H 1995 Annihilation poles of a Smirnov-type integral formula for solutions to the quantum Knizhnik–Zamolodchikov equation *J. Phys. A: Math. Gen.* **28** 3479–91

[2] Frenkel I B and Reshetikhin N Yu 1992 Quantum affine algebras and holonomic difference equations *Commun. Math. Phys.* **146** 1–60

[3] Smirnov F A 1992 *Form Factors in Completely Integrable Models of Quantum Field Theory* (Singapore: World Scientific)

[4] Smirnov F A 1995 Counting the local fields in sine-Gordon theory *Nucl. Phys.* B **453** 807–24

[5] Smirnov F A 1996 On the deformation of Abelian integral *Lett. Math. Phys.* **36** 267–75

[6] Babelon O, Bernard D and Smirnov F A 1996 Quantization of solitons and the restricted sine-Gordon model *Commun. Math. Phys.* to appear

[7] Babelon O, Bernard D and Smirnov F A 1996 Null-vectors in integrable field theory *Preprint* hep-th/9606068

[8] Davies B, Foda O, Jimbo M, Miwa T and Nakayashiki A 1993 Diagonalization of the XXZ Hamiltonian by vertex operators *Commun. Math. Phys.* **151** 89–153

[9] Jimbo M and Miwa T 1994 *Algebraic analysis of solvable lattice models* CBMS Regional Conferences Series in Mathematics

[10] Jimbo M, Kojima T, Miwa T and Quano Y-H 1994 Smirnov's integrals and quantum Knizhnik–Zamolodchikov equation of level 0 *J. Phys. A: Math. Gen.* **27** 3267–83

[11] See e.g. Jimbo M 1992 Topics from representations of $U_q(\mathfrak{g})$—an introductory guide to physicists *Nankai Lectures on Mathematical Physics* (Singapore: World Scientific) pp 1–61